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2001 J. Phys. A: Math. Gen. 34 8517

(http://iopscience.iop.org/0305-4470/34/41/309)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 8517–8529 PII: S0305-4470(01)14717-7

Optical polarization and symmetry invariances

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Received 8 June 2000, in final form 25 June 2001
Published 5 October 2001
Online at stacks.iop.org/JPhysA/34/8517

Abstract
In this paper, we provide a mathematically sound description of the propagation of interacting polarized radiation. It appears that some peculiar group invariants can successfully describe the features of the propagation of the two transverse polarization states through anisotropic media.

PACS number: 42.25.Ja

1. Introduction

Since the discovery in 1964 of the *CP* violation in $K_L \to \pi^+\pi^-$ decays [1], many proposals were undertaken to experimentally observe this phenomenon in other physical processes. However, apart from the result of the Collaboration CPLEAR at CERN in $K_{\ell 3}$ decays [1], no experiment has been completely successful in detecting any further violation of the discrete sector of the relativistic invariances. At present, some proposals are actively discussed to evidence *T*-violating effects in the propagation of the laser light in an optically active atomic or molecular gas placed in an electromagnetic field [2]. The violation of the time-reversal symmetry can be associated with the manifestation of the electric–dipole momentum not only of atoms and molecules but also of the nucleons composing the traversed matter. However, this fundamental description of the propagation of electromagnetic radiation in a medium is obscured by several problems, which are simply an expression of our ignorance about the dynamics of all processes involved in the medium. The microscopic approach for introducing the propagation of radiation requires the knowledge of complete interactions between radiation and matter and the possibility to integrate, in some approximation, the matter degrees of freedom. This procedure can hardly be performed for arbitrary media, and therefore only simplified examples are considered [3]. From a purely macroscopic point of view, the crucial question is to append certain *constitutive relations* and to feed them into the Maxwell equations. The presence of induced sources is taken into account by introducing some empirical constants in the defining equations. The number of such constants depends on the nature of the medium.

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For linear isotropic materials, it is well known that one needs two scalar quantities, the dielectric permittivity ε and the magnetic permeability μ, to describe all phenomena of macroscopic electrodynamics, according to the relations

$$\vec{D} \equiv \varepsilon_0 \vec{E} + \vec{P} = \varepsilon \vec{E} \qquad \vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} = \mu^{-1} \vec{B}. \tag{1}$$

Once the empirical constants ε and μ are known, one can then derive the electric vector field $\vec{E}$ and magnetic vector field $\vec{B}$. However, for complex systems, one needs in general more than these two constants to describe electrodynamical phenomena in them [4]. In many instances, this inadequacy may be addressed by postulating a more general set of constitutive relations with a larger number of electromagnetic constants, depending on the nature of the medium in question. In chiral media, which are special bi-isotropic linear media, they are rank-2 tensors. However, it is clear that in a real material, microscopic causality forces the macroscopic constitutive relations to involve convolutions. Although the approach of postulating general constitutive relations is well known, only recently has it been discussed in connection with the symmetry requirements which could be of great help in analysing the dynamics of the propagation [5]. It is, however, known that group theory can describe the propagation of the two transverse modes of the electromagnetic radiation. In the context of the relativistic quantum theory of elementary systems, the internal symmetries of photons are usually described by means of the light-like finite helicity unitary irreducible representations of the Poincarè group. Obviously, these fundamental features of the photon in vacuum should not change when we consider its propagation in a medium. However, the inclusion of an interacting background can drastically change its physical behaviour and then the description of the polarization states of a single elementary photon in a medium. Its Poincarè invariant dynamics should be enlarged to include the non-degenerate behaviour of the two transverse modes of polarization when it propagates through anisotropic or active media and also when an external background exerts a selective action on one of them, as in the case of the Faraday effect. In this paper, we propose to re-assert the group theory of the photon propagation in linear anisotropic media with spatially varying permittivity and the connected intriguing features of the symmetry violations. An important singularity of this paper is the inclusion of the absorptive effects and the enlargement of the standard approach which is based on the rotation group representations. After the introduction of a shorthand notation, in section 3, we describe the photon propagation in linear birefringent media by means of their group invariants. In a further section, we evidence the complications of a naive helicity description of the polarization observables for the propagation in dichroic materials. The aim of this paper is to outline the importance of the induced symmetry breaking features due to matter background and, at the same time, to examine the difficulties of co-existence with a massless theory of the photon propagation.

2. Electromagnetic radiation in a medium

Many properties of the electromagnetic fields may be better understood if one considers the Fourier-space version of the Maxwell equations. In fact, in many fields of physics, including electrodynamics, it is convenient to rewrite the (Maxwell) differential equations of motion in terms of their Fourier components. In this linear approximation, from the usual results of Fourier analysis, we express a general solution as a superposition of plane waves through the relation

$$\vec{E}(t, \vec{x}) = \frac{1}{(2\pi)^2} \int \mathrm{d}\omega \int \mathrm{d}^3\vec{k}\, \tilde{\tilde{\vec{E}}}(\omega, \vec{k})\, \exp[\mathrm{i}(\vec{k} \cdot \vec{x} - \omega t)] \tag{2}$$

and similarly for other quantities. Now, if the Fourier transform of a linear equation is taken, the space and time drop out. By these means, the set of coupled (Maxwell) differential equations is replaced by a set of coupled algebraic equations, which of course greatly reduces the difficulty. The solutions of the algebraic equations are the Fourier transforms of the required quantities which can then be recovered by means of a Fourier inversion. The Fourier momentum space version of the Maxwell equations in a medium is then given by

$$\vec{k}\cdot\tilde{\tilde{\vec{B}}}=0 \qquad \vec{K}\times\tilde{\tilde{\vec{E}}}=\omega\tilde{\tilde{\vec{B}}} \tag{3}$$

$$\mathrm{i}\vec{k}\cdot\tilde{\tilde{\vec{D}}}=\tilde{\rho}_{\mathrm{ext}} \qquad \mathrm{i}\vec{k}\times\tilde{\tilde{\vec{B}}}=\mu_0\tilde{\tilde{\vec{J}}}-\mathrm{i}\frac{\omega}{c^2}\tilde{\tilde{\vec{E}}} \tag{4}$$

with source terms of just the external charges and currents. In writing these relations it is assumed that the response of the system to the applied electromagnetic fields is linear. Even when nonlinear effects are considered, they could be usually treated as corrections to linear ones. In this approach, it is assumed that all quantities which vary with time and space differ only slightly from an average value. The source Maxwell equations reduce to the wave form

$$\frac{\omega^2}{c^2}\tilde{\tilde{\vec{E}}}+\vec{k}\times(\vec{k}\times\tilde{\tilde{\vec{E}}})=-\mathrm{i}\omega\mu_0\tilde{\tilde{\vec{J}}} \tag{5}$$

so that we can write [6]

$$\tilde{\tilde{\vec{E}}}_i=\tilde{D}_{ij}(\tilde{\tilde{\vec{J}}}_{\mathrm{ext}})_j \tag{6}$$

by means of the elimination of the induced part $\tilde{\tilde{\vec{J}}}_{\mathrm{ind}}$ of the current $\tilde{\tilde{\vec{J}}}=\tilde{\tilde{\vec{J}}}_{\mathrm{ind}}+\tilde{\tilde{\vec{J}}}_{\mathrm{ext}}$, and with the inverse Green function given by

$$\tilde{D}_{ij}^{-1}(\omega,\vec{k})=\frac{c^2}{\omega^2}\left(k_ik_j-k^2\delta_{ij}\right)+\tilde{K}_{ij}(\omega,\vec{k}) \tag{7}$$

assuming the equivalent response tensor

$$\tilde{K}_{ij}(\omega,\vec{k})=\delta_{ij}+\chi_{ij}. \tag{8}$$

Here, we propose to study non-magnetic but otherwise arbitrary linear media which can be described in terms of spatially varying permittivity. The difficult separation of the induced current into electric and magnetic parts,

$$\tilde{\tilde{\vec{J}}}_{\mathrm{ind}}=-\mathrm{i}\omega\tilde{\tilde{\vec{P}}}+\mathrm{i}\vec{k}\times\tilde{\tilde{\vec{M}}} \tag{9}$$

and eventually into other parts due to extra interactions leads to confusion. We include in equation (6) all the responses of the medium. If the medium is dispersionless and lossless, $\tilde{K}_{ij}$ is real symmetric. For dispersive and absorbing dielectric media, the tensor $\tilde{K}_{ij}$ may be separated into a Hermitian part and an anti-Hermitian part,

$$\tilde{K}_{ij}=\tilde{K}_{ij}^{\mathrm{H}}+\tilde{K}_{ij}^{\mathrm{A}} \tag{10}$$

with

$$\tilde{K}_{ij}^{\mathrm{H}}=\tfrac{1}{2}(\tilde{K}_{ij}+\tilde{K}_{ij}^{*}) \tag{11}$$

$$\tilde{K}_{ij}^{\mathrm{A}}=\tfrac{1}{2}(\tilde{K}_{ij}-\tilde{K}_{ij}^{*}). \tag{12}$$

Dispersion and absorption are related by the Kramers–Kronig relations which assure that these components are not independent. In fact,

$$\tilde{K}_{ij}^{\mathrm{A}}=\frac{\mathrm{i}}{\pi}\mathcal{P}\int_{-\infty}^{+\infty}\frac{\tilde{K}_{ij}^{\mathrm{H}}-\delta_{ij}}{\omega-\omega'}\,\mathrm{d}\omega' \tag{13}$$

$$\tilde{K}_{ij}^{\mathrm{H}}-\delta_{ij}=\frac{\mathrm{i}}{\pi}\mathcal{P}\int_{-\infty}^{+\infty}\frac{\tilde{K}_{ij}^{\mathrm{A}}}{\omega-\omega'}\,\mathrm{d}\omega'. \tag{14}$$

For an inhomogeneous medium, that is spatially dispersive, we have three independent (transverse, longitudinal and rotatory) parts in K_{ij}^{H},

$$\tilde{K}_{ij}^{\mathrm{H}} = \varepsilon^{\mathrm{T}}(\delta_{ij} - \hat{\vec{k}}_i\hat{\vec{k}}_j) + \varepsilon^{\mathrm{L}}\hat{\vec{k}}_i\hat{\vec{k}}_j + \varepsilon^{\mathrm{R}}\epsilon_{ijr}\hat{\vec{k}}_r = \begin{pmatrix} \varepsilon^{\mathrm{T}} & \varepsilon^{\mathrm{R}} & 0 \\ -\varepsilon^{\mathrm{R}} & \varepsilon^{\mathrm{T}} & 0 \\ 0 & 0 & \varepsilon^{\mathrm{L}} \end{pmatrix} \tag{15}$$

where the last matrix form holds, if we choose $\vec{k}$ along the z-axis. The fields $\tilde{\vec{E}}$ and $\tilde{\vec{B}}$ depend on $(\omega, \vec{k})$, whereas the quantities ε^{T}, ε^{L} and ε^{R} characterize the properties of the medium and are supposed to depend on $\vec{k}$ only through its magnitude $k \equiv |\vec{k}|$. In fact we are considering a medium in which only the amplitudes of the electric and magnetic polarizations depend on direction. The electromagnetic constants ε^{T}, ε^{L} and ε^{R} characterize directly any anisotropic effect. Before we proceed further, we want to note that there is one relation that ε^{T}, ε^{L} and ε^{R} must satisfy on very general grounds. It is necessary that the fields $\vec{E}$, $\vec{B}$ and the current $\vec{J}$ are, in coordinate space, real quantities. This yields that, for real values of ω, the electromagnetic constants satisfy the crossing prescription

$$\varepsilon^*(-\omega, k) = \varepsilon(\omega, k) \tag{16}$$

where ε refers to the electromagnetic constants ε^{T}, ε^{L} or ε^{R} and the dependence is supposed to be through its magnitude k, as noted earlier. In case of dissipative media, the response tensor components will follow the Onsager relations

$$\begin{aligned} \tilde{K}_{ij}^{\mathrm{H}}(-\omega, \vec{k}) &= \tilde{K}_{ij}^{\mathrm{H}}(\omega, \vec{k}) \\ \tilde{K}_{ij}^{\mathrm{A}}(-\omega, \vec{k}) &= -\tilde{K}_{ij}^{\mathrm{A}}(\omega, \vec{k}) \end{aligned} \tag{17}$$

whereas in general $\tilde{K}_{ij}(\omega, -\vec{k}) = \tilde{K}_{ij}(\omega, \vec{k})$.

From the above relations, it is quite clear that the presence of the ε^{R}-term is expected to vanish unless complicated processes (including (super) weak interations) take place within the medium.

3. The propagator formalism

In this section, we consider the general description of the propagation of a polarized state in a medium. The nature of the wave propagation is, in general, studied from some dispersion relations. These are the relations that ω and $\vec{k}$ must satisfy in order to solve the homogeneous wave equation for the $\vec{E}$ and $\vec{B}$ fields. In a compact notation, which includes the inverse Green function matrix propagator, we can write

$$\tilde{D}_{ij}^{-1}\tilde{E}_j = 0. \tag{18}$$

In a complex medium, the existence of a solution is connected to the dispersion relation

$$\det \tilde{D}_{ij}^{-1} = 0. \tag{19}$$

When this dispersion relation is satisfied, there exists a solution of the homogeneous wave equation. In the vacuum one can discuss the transverse solution,

$$\tilde{\vec{E}}_{\perp} = \tilde{E}_1\hat{\vec{\epsilon}}_1 + \tilde{E}_2\hat{\vec{\epsilon}}_2 \tag{20}$$

in terms of the components,

$$\tilde{E}_1 = \tilde{E}_x \qquad \tilde{E}_2 = \tilde{E}_y \tag{21}$$

of the electric field transverse to the coherent beam propagating in the $\hat{\vec{k}} = \hat{\vec{z}}$ direction. In this description of polarization, we refer to the two basic polarization orthonormal unit vectors $\hat{\vec{\epsilon}}_1$ and $\hat{\vec{\epsilon}}_2$, both being perpendicular to the wave unit vector $\hat{\vec{k}}$. An equivalent set of basic vectors is given by the following right- and left-circular basis:

$$\begin{aligned} \hat{\vec{\epsilon}}_+ &= \frac{1}{\sqrt{2}}(\hat{\vec{\epsilon}}_1 + \mathrm{i}\hat{\vec{\epsilon}}_2) \\ \hat{\vec{\epsilon}}_- &= \frac{1}{\sqrt{2}}(\hat{\vec{\epsilon}}_1 - \mathrm{i}\hat{\vec{\epsilon}}_2). \end{aligned} \tag{22}$$

In the vacuum, the two transverse modes have the same physical characteristics, with the same speed of propagation c, and there would also be no difference in the physical properties of right- and left-circular polarization states of the transverse electromagnetic wave. In an isotropic homogeneous medium (when $\varepsilon^{\mathrm{R}} = 0$), far from external perturbations, this feature is preserved, the velocity is modified to c/n due to the presence of interacting fluctuations and the response tensor is given by

$$\tilde{K}_{ij} = \varepsilon^{\mathrm{L}}\hat{\vec{k}}_i\hat{\vec{k}}_j + \varepsilon^{\mathrm{T}}(\delta_{ij} - \hat{\vec{k}}_i\hat{\vec{k}}_j). \tag{23}$$

In this case, the dispersion relation

$$\det \tilde{D}_{ij}^{-1} = \det\left[\varepsilon^{\mathrm{L}}\hat{\vec{k}}_i\hat{\vec{k}}_j - (\varepsilon^{\mathrm{T}} - n^2)(\hat{\vec{k}}_i\hat{\vec{k}}_j - \delta_{ij})\right] = \varepsilon^{\mathrm{L}}(\varepsilon^{\mathrm{T}} - n^2)^2 = 0 \tag{24}$$

yields a longitudinal relation

$$\varepsilon^{\mathrm{L}}(\omega, k) = 0 \tag{25}$$

and a transverse dispersive equation

$$n^2 = \varepsilon^{\mathrm{T}}(\omega, k) \tag{26}$$

which characterizes the speed of the two degenerate modes. Whether dissipative effects are considered or not, the two transverse modes no longer have the same dispersion relation. The transverse components $\tilde{\vec{E}}_\perp$ are governed in their evolution by the following inverse matrix propagator:

$$(\tilde{D}^\perp)_{ij}^{-1} = -\frac{k^2c^2}{\omega^2}\delta_{ij}^\perp + \left(\delta_{ij}^\perp + \chi_{ij}^\perp\right) \tag{27}$$

where the symbol $\perp$ refers to the transverse sector. In isotropic optically active media ($\varepsilon^{\mathrm{R}} \neq 0$), parity, and, to a lesser extent, time-reversal are broken. It is evident that these two transverse modes have different properties when ε^{R} does not vanish. The model of dissipation will cause the non-trivial complex form of $\varepsilon^{\mathrm{R}}(\omega, k)$. Neglecting dissipation, but including optically active backgrounds, it becomes purely imaginary. In this case, for a large range of ε^{R}, the dispersion relation is solved for real ω and k. Thus, the amplitude of the wave will remain unchanged during the propagation, which means that there is no absorption in the medium. The problem of evolution of the two transverse polarization states of the electromagnetic radiation propagating in a complex medium can be described in many ways, nevertheless, a dispersion relation governing the propagation is required [6].

4. The linear optical polarization

In this section, we formulate the group theory description of the possible polarization states of a photon traversing matter by means of the relevant invariants. This issue was already considered in the literature, but in some reduced analogies and with an obsolete formalism [7].

Now, we study a coherent polarized light beam that is incident on a medium at $z = 0$ and propagating in the positive z-direction. The transverse electric components can be assumed in a normal form of the type $\vec{E}_\perp(z, t) = \vec{E}_\perp(z)\mathrm{e}^{-\mathrm{i}\omega t}$, where

$$\vec{E}_\perp(z) = E_1(z)\hat{\vec{\epsilon}}_1 + E_2(z)\hat{\vec{\epsilon}}_2. \tag{28}$$

The space propagation of the transverse optical polarization states is governed by the transverse response tensor

$$K_{ij}^\perp = \varepsilon\mu\left(\frac{\omega}{c}\right)^2\left[\delta_{ij} + \chi_{ij}^\perp\right] = \varepsilon\mu\left(\frac{\omega}{c}\right)^2\tilde{\mathcal{N}}^2 \tag{29}$$

which is given in terms of a general non-Hermitian susceptibility tensor

$$\chi^\perp = \begin{pmatrix} \chi_{11} & \chi_{12} \\ \chi_{21} & \chi_{22} \end{pmatrix}. \tag{30}$$

When the light beam propagates through the medium, the absorption coefficient in one transverse direction could be different from the coefficient along another direction. However, we can rewrite the transverse components of the electric field in a two-component Jones vector, whose space evolution is given according to the relation

$$\begin{pmatrix} E_1(z) \\ E_2(z) \end{pmatrix} = \exp[-\mathrm{i}z\tilde{\mathcal{N}}]\begin{pmatrix} E_1(0) \\ E_2(0) \end{pmatrix} \tag{31}$$

where

$$\tilde{\mathcal{N}} = \frac{\omega\sqrt{\epsilon\mu}}{c}\begin{pmatrix} 1+\chi_{11} & \chi_{12} \\ \chi_{21} & 1+\chi_{22} \end{pmatrix}^{1/2}. \tag{32}$$

The eigenvalues of the square-root matrix $\tilde{\mathcal{N}}$ are

$$\tilde{n}_{1,2} = \tfrac{1}{2}(\Sigma \pm \Delta) \tag{33}$$

with

$$\begin{aligned} \Sigma &= \mathrm{tr}\,\tilde{\mathcal{N}} \\ \Delta^2 &= \mathrm{tr}^2\,\tilde{\mathcal{N}} - 4\det\tilde{\mathcal{N}}. \end{aligned} \tag{34}$$

Thus, two different (complex) indices of refraction, $\tilde{n}_1$ and $\tilde{n}_2$, characterize the medium when the complex tensor $\tilde{\mathcal{N}}_{ij}$ is referred to its principal axes. If we consider the Jones vector representation, we obtain a two-by-two transverse formalism. In the case of linear birefringence, where the absorption coefficient in one principal transverse direction is different from the coefficient in the other direction, we can write

$$\begin{pmatrix} E_1'(z) \\ E_2'(z) \end{pmatrix} = \mathrm{e}^{-\frac{1}{2}(\eta_1+\eta_2)(z-z_0)}S(0,\eta)\begin{pmatrix} E_1'(z_0) \\ E_2'(z_0) \end{pmatrix} \tag{35}$$

where $\eta_{1,2} = (\mathrm{i}\omega/c)\tilde{n}_{1,2}$ and $\eta = \eta_2 - \eta_1$. In this relation we consider the space evolution of the polarized components between two positions z_0 and z. The exponential factor $\mathrm{e}^{-1/2(\eta_1+\eta_2)}$ reduces both components at the same rate and does not affect the degree of polarization. The anisotropic attenuation effect is determined by the matrix

$$S(0,\eta) = \begin{pmatrix} \mathrm{e}^{\eta/2} & 0 \\ 0 & \mathrm{e}^{-\eta/2} \end{pmatrix}. \tag{36}$$

In the most generic case, the polarization plane is inclined by an angle $\theta/2$, and the field components are rotated by the rotation matrix:

$$R(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}. \tag{37}$$

In this case, the attenuator matrix becomes

$$S(\theta,\eta) = R(\theta)S(0,\eta)R(-\theta). \tag{38}$$

This transforming matrix can be represented by means of the generators,

$$K_1 = -\tfrac{\mathrm{i}}{2}\sigma_3 \tag{39}$$

$$K_2 = -\tfrac{\mathrm{i}}{2}\sigma_1 \qquad \text{but} \quad K_3 = -\tfrac{1}{2}\sigma_2 \tag{40}$$

of the homogeneous symplectic group $Sp(2)$ which satisfy the following set of closed commutation relations:

$$\begin{aligned}
&[K_1, K_2] = -\mathrm{i}K_3\\
&[K_1, K_3] = -\mathrm{i}K_2\\
&[K_2, K_3] = \mathrm{i}K_1.
\end{aligned} \tag{41}$$

In this formalism, the attenuator matrix becomes

$$S(0,\eta) = \exp(-\mathrm{i}\eta K_1) \tag{42}$$

and

$$S(\theta,\eta) = \exp(aK_1 + bK_2) \tag{43}$$

if a rotation is performed, with

$$a = \eta\cos\theta \qquad b = \eta\sin\theta. \tag{44}$$

Therefore, the transformation matrix for the two-component (Jones) vector can be written as

$$\begin{aligned}
S(\theta,\eta) &= \begin{pmatrix} \cosh(\eta/2) + \sinh(\eta/2)\cos\theta & \sinh(\eta/2)\sin\theta \\ \sinh(\eta/2)\sin\theta & \cosh(\eta/2) - \sinh(\eta/2)\cos\theta \end{pmatrix}\\
&\simeq I\cosh\frac{\eta}{2} - \mathrm{i}(2K_1\cos\theta + 2K_2\sin\theta)\sinh\frac{\eta}{2}.
\end{aligned} \tag{45}$$

This transformation matrix has already been proposed and extensively discussed in connection with the group representation of coupled oscillating systems [8]. The results of these applications on polarization optics have been partially noted in the literature [7, 9]. In this context, it has been noted that the group theory realization of optical devices, which modify the polarized state of the radiation, is usually treated as realizing the action of the $SU(2)$ (or $U(2)$) group representations on the two-dimensional space of the transverse polarization states. Now, we propose to consider properly the transformation properties of the polarization observables and list all those quantities which are invariant under different symmetries. This provides a firm and mathematically sound foundation for certain informal uses of polarization variables. Usually the polarization states of a massless field are described by the inhomogeneous Lorentz dynamics. The action of the optical devices are thus connected to the $SU(2)$ transformations acting in the two-dimensional space of the transverse polarization states. Our approach can shed light on the physical nature of these transformations. We must note that the characterization of the polarization states of a plane electromagnetic wave made essential use of the Poincaré sphere representation [10]. In the context of our comprehensive notation, we must refer to the two-dimensional subspace orthogonal to the propagating wave vector $\hat{\vec{k}}$ (we suppose along the $\hat{z}$-axis), which is a plane tangent to a unit sphere. The orientation of the local $\hat{x}$- and $\hat{y}$-axes can be chosen consistently to span the tangent plane. The electric field components E_x and E_y are transverse to $\hat{\vec{k}}$ and conventionally are represented by a two-component Jones vector. The salient features of the Poincaré sphere representation can be recalled by using them throughout the notation and terminology of quantum mechanics. We deal with a two-dimensional complex Hilbert space, whose components are the Jones vectors.

Now, it becomes clear that the degree and type of the polarization of the resulting transverse electric wave are associated with a unit vector and therefore with a point on the unit sphere [11]. The stereographic projection of this unit vector onto the equatorial plane can be consequently associated with the complex components of a Poincaré spinor [10]. This is nothing else but the usual omorphism between the $SO(3)$ rotation group and its covering group $SU(2)$. To make clearer this correspondence, it is worth noting that the two-component Poincaré spinor is thus associated with a vector representing the angular momentum for a spin-$\frac{1}{2}$ state. This vector is just the generator of the internal rotations [12]. Its components give the Cartesian components of the polarized electric field. However, the representation of the electric field $\vec{E}$ in two-dimensional $SU(2)$ complex space has some difference with the $SO(3)$ vector realization in a real space. The ϕ rotation in the real $SO(3)$ space corresponds to a $\phi/2$ rotation in the $SU(2)$ space, due to the two-to-one correspondence of their parameter spaces. To make this statement significant, we observe that a rotation of the polar field representation

$$\vec{E} = \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix} \tag{46}$$

can be associated with a two-dimensional unitary matrix

$$U = \begin{pmatrix} U_{11} & -U_{21}^{\star} \\ U_{21} & U_{11}^{\star} \end{pmatrix} \tag{47}$$

by means of the Hopf map

$$\vec{E} = \pi_{\mathrm{H}}(U) = \begin{pmatrix} 2\,\mathrm{Re}(U_{11}U_{21}^{\star}) \\ 2\,\mathrm{Im}(U_{11}U_{21}^{\star}) \\ |U_{11}|^2 - |U_{21}|^2 \end{pmatrix} = (U_{11}^{\star}\, U_{21}^{\star})\vec{\sigma} \begin{pmatrix} U_{11} \\ U_{21} \end{pmatrix} \tag{48}$$

up to an uncertainty of a phase $\alpha = U_{11}/|U_{11}|$. The phase uncertainty degree of freedom is usually lost upon passing to intensities. Incidentally, we notice that this 2×2 unimodular matrix representation of the Pauli algebra to describe the polarization can be replaced by the 4×4 Mueller matrix method [13]. Further, another modern method using 2×2 matrices to represent polarization is represented by the Fano density operator approach [14] for the transverse electric field components, analogously to the case of spin density theory for the spin-$\frac{1}{2}$ particles. It is worth noting that while the fundamental mathematical descriptions of spin $\frac{1}{2}$ and optical polarization are practically the same, their physical interpretations are quite different. All this can be understood if we realize that any polarization observable is based on the density matrix

$$J_{ij} = E_i E_j^{*}. \tag{49}$$

After providing a computational framework, we can expand $\vec{J}$ in terms of the Pauli matrices according to the expression

$$\vec{J} = \tfrac{1}{2}(1 + \hat{\hat{n}}\cdot\vec{\sigma}) = \tfrac{1}{2}(1 + \hat{\not{n}}) \tag{50}$$

where the maximal degree of polarization is assumed unitary, and we have introduced the unit 3-vector

$$\hat{\hat{n}} = \hat{\hat{n}}^{*} \qquad \hat{\hat{n}}\cdot\hat{\hat{n}} = 1 \qquad \text{with} \quad \hat{\hat{n}} \in S^2. \tag{51}$$

Here $\vec{\sigma}$ are the standard Pauli matrices and $S^2 = SU(2)/U(1)$ is the two-dimensional sphere. We can now consider the 'Poincaré' coordinates θ, χ of the unit vector $\hat{\hat{n}}$, and consequently, we observe that the orientation of a linear polarized electric field is labelled by a polarization angle χ in the spherical-polar system of coordinates. Note that this representation does not refer to vectors in the rigorous sense, because a rotation of π radians returns an axis to its original orientation. In fact, we are aware of a real projective group. In other words, adopting a 'slash' notation for the contraction of a vector with the Pauli matrix,

$$\hat{\not{n}} = \hat{\vec{n}} \cdot \vec{\sigma} = \begin{pmatrix} \cos\theta & \sin\theta\, \mathrm{e}^{2\mathrm{i}\chi} \\ \sin\theta\, \mathrm{e}^{-2\mathrm{i}\chi} & -\cos\theta \end{pmatrix} \tag{52}$$

we can see that each pure polarization state corresponds to a point on the two-dimensional unit sphere S^2 embedded in Euclidean three-dimensional space $\mathbb{R}^3$. Considering

$$\begin{aligned} \vec{J} &= \langle E_i | E_j \rangle = \tfrac{1}{2}(1 + \hat{\vec{n}} \cdot \vec{\sigma}) \\ \vec{J}' &= \langle E_i' | E_j' \rangle = \tfrac{1}{2}(1 + \hat{\vec{n}}' \cdot \vec{\sigma}) \end{aligned} \tag{53}$$

we can write the parametric evolution of a polarized state by means of the following fundamental relation:

$$\mathrm{Tr}(\vec{J}\vec{J}') = |\langle E' | E \rangle|^2 = \tfrac{1}{2}(1 + \hat{\vec{n}}' \cdot \hat{\vec{n}}). \tag{54}$$

In this expression, it is manifested that diametrically opposite points on the Poincaré sphere S^2 correspond to mutually orthogonal polarization states. If a polarization state $|E\rangle$ is subjected to a unitary transformation $U \in SU(2)$, the representative point $\hat{\vec{n}} \in S^2$ undergoes an orthogonal rotation $R(U)$ belonging to $SO(3)$, i.e.

$$\hat{n}_j' = R_{jk}(U)\hat{n}_k \qquad \text{with} \qquad R_{jk}(U) = \tfrac{1}{2}\,\mathrm{Tr}(\sigma_j U \sigma_k U^\dagger) \tag{55}$$

or, in other words, $\hat{\not{n}}' = U\, \hat{\not{n}} U^\dagger$. Since all elements $R \in SO(3)$ are realized in this way, we have the trivial coset space identification $S^2 = SU(2)/U(1) = SO(3)/SO(2)$. This result shows that the rotations in the spin-$\frac{1}{2}$ representation are governed by the same rule of the vector rotations.

5. The circular optical polarization

From the point of view of group theory, the relevant unitary irreducible representations should lead only to a single polarization state, with helicity ± 1 corresponding to right- or left-circular polarization, respectively. Helicity is expected relativistically invariant, and only the action of the parity operator P can connect states of opposite helicities. Rather interestingly, the split-up of the three-dimensional orthogonal group into a disconnected subgroup of different parity suggests to look for new observables connected with the circular polarization. It should be intuitively clear that the emergence of parity as an observable would allow one to extract a definition of right- versus left-handed sense of rotation. In the circular polarization basis the electric field will propagate as

$$\begin{aligned} E_+(z) &= E_+(0)\mathrm{e}^{-\mathrm{i}\tilde{n}_+ z} \\ E_-(z) &= E_-(0)\mathrm{e}^{-\mathrm{i}\tilde{n}_- z}. \end{aligned} \tag{56}$$

The effect of a dichroic material, i.e. a medium with different absorption in the two circular polarization directions can be described as

$$\begin{pmatrix} E_+'(z) \\ E_-'(z) \end{pmatrix} = \mathrm{e}^{-\frac{1}{2}(\lambda_+ + \lambda_-)(z - z_0)} P(0, \lambda) \begin{pmatrix} E_+'(z_0) \\ E_-'(z_0) \end{pmatrix} \tag{57}$$

in terms of $\lambda_\pm = (\mathrm{i}\omega/c)\tilde{n}_\pm$ and $\lambda = \lambda_- - \lambda_+$. The matrix

$$P(0, \lambda) = \begin{pmatrix} \mathrm{e}^{\lambda/2} & 0 \\ 0 & \mathrm{e}^{-\lambda/2} \end{pmatrix} \tag{58}$$

characterizes the different propagation in the left- and right-handed sense of rotation. If there is an angle $\phi/2$ between the rotating fields $E'_{\pm}$, the general form of the matrix P is given by

$$P(\phi,\lambda) = R(\phi)P(0,\lambda)R(-\phi) \simeq I\cosh\frac{\lambda}{2} - \mathrm{i}(2K_1\cos\phi + 2K_2\sin\phi)\sinh\frac{\lambda}{2} \tag{59}$$

where we have used the generators of the group $SU(2)$ defined by the relations (39) and (40).

This conclusion becomes relevant if one observes that only extremal helicities survive for massless particles of any spin. This result is intuitively quite natural since massless unitary representations of the Poincaré group are generated by a one-dimensional representation of the Euclidean subgroup $E(2)$ [15]. Typically, the Euclidean structure of massless fields corresponds to the localization of the massless particles in a plane perpendicular to their momentum. The vanishing of the eigenvalues of spin in a direction perpendicular to the momentum can be interpreted as a consequence of the Lorentz flattening of the moving particles. The algebraic curiosity that the components of the spin satisfy an algebra which is $SO(3)$ for a massive particle in its rest frame and formally contract to the Euclidean $E(2)$ little group, in the massless case, imposes a clear definition and an alternative explanation of the two-component position operator of the Maxwell field. Moreover, the privileged role played by the Euclidean group in the theory of massless fields and the fact that the representation space is topologically non-trivial prevent the possibility to separate space from spin variables. The $E(2)$ group contains one rotational and two 'translational' degrees of freedom. The rotational degree of freedom is associated with the photon spin which is either parallel or anti-parallel to the photon momentum. The nature of the 'translational' degree of freedom has many possible interpretations. It can be shown that the generators of the translations can generate gauge transformations [16]. On the other hand, the $E(2)$ structure is associated with a cylindrical geometry. The Euclidean group contains rotations around and translations along a cylinder which is parallel to the momentum [17]. It is worth noting that the Lorentz flattening of a massless particle occurs also naturally in the twistor formalism [18]. A twistor is a kind of 'square root' of generators of the Poincaré group on the light cone and is associated with the representations of the conformal group. It is known that although spin-0 twistors can be represented geometrically by null straight lines, this does not hold for spin-j, $j \neq 0$, twistors. Instead of the straight line we get a congruence of twisting null, shear-free world lines, the so-called Robinson congruence. A three-dimensional projection of this congruence consists of circles which propagate with light velocity in the momentum direction and rotate in the right- or left-hand sense depending on the sign of helicity. The Robinson congruence picture suggests that classical massless fields may be related naturally to circular strings whose radii are different for different observers. This string-like picture of massless fields is however even deeper and can be related to the old problem of the localization of spinning particles. Obviously, these fundamental features of the photon in vacuum do not change whenever we consider its propagation into a medium. However, the inclusion of an interacting background can drastically change its physical behaviour, and then its description. As far as the description of the polarization states of a single elementary photon in a medium, the Poincaré invariant dynamics should be enlarged. The development of this approach leads to an unavoidable momentum dependence of all physically important polarization observables. At a purely formal level, the practical description of the polarization states of massless photons in a medium has more than one analogy with the meaning of the mass of an interacting field. In this case, the difficulties of a reformulation of the (non)unitary representations of the Poincaré group are connected with the replacement of the concrete on-shell mass with an off-shell pole [19]. This formal difficulty is related to the attempt to characterize the propagating particles in the context of the group representation theory, which may be connected to a complex rest mass [20]. On the other hand, these features cannot be extended naturally to the massless case, just because the Euclidean nature of the little group is associated with the massless particles. The extension with a strictly massless particle deserves many complications due to intricacies with domains of unbounded operators. From the point of view of an extended group representation theory, we can include the influence of the medium on the photon definition. The representation group theory which includes interactions is essentially a bi-local (i.e. p-dependent) $SU(2)$ theory. The theory for relativistic systems with internal structure is not physically well founded. The massless constraint $p \cdot p \to 0\,(p \neq 0)$ imposes the introduction of a two-dimensional space of polarization states whose elements realize a local $SU(2)$ Lie algebra on this space [21]. All physically observables can be built up of the generators of $SU(2)$ acting on the polarization states of the photon at each fixed energy momentum. It is worth stressing that there is an essential momentum dependence in these constructions, which cannot be eliminated globally. It emerges a Poincaré invariant formalism for a two $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ neutrinos theory of photons [22]. The formulation of particle–anti-particle systems, also in the case where the compounds are massless fermion particles, requires a detailed study from a mathematical point of view. However, it seems that the physical interpretation is rather confusing, since it is not easy to outline the covariant form of the possible Hamiltonian dynamics.

6. Discussion and conclusions

The main feature of the transverse electromagnetic waves propagating in a medium consists in the possible split of its two normal modes. The two transverse polarized components become non-degenerate when they propagate through anisotropic media or also when an external background can exert a selective action on one of them. This is the case of the Faraday effect, which arises as a consequence of the induced coupling between the traversing radiation and a pure magnetic external field [23]. As it is shown, an exhaustive theory of the propagation of these two modes can be addressed with the group representations. Up to now, experimental investigations seem indifferent to all these mathematical complications. In fact, the modification of the photon dispersion relations turns out to be very small for the strengths of any external influence presently attainable. Nevertheless the outlined complications could be of relevance for the physics of neutron stars, supernovas and white dwarfs [24], and may be observable in the laboratory due to the recent developments in laser technology [25]. In principle, our study could also provide an explanation of the recent finding of the anisotropic propagation of the cosmic background radiation claimed by Nodland and Ralston [26]. On the other hand, the theory of the birefringent effects in the propagation of the electromagnetic radiation in crystals [27], unfortunately, does not find wide experimental counterparts. No experiment has been performed to date to corroborate the main features of a photon beam with accelerator energies passing through crystals despite some relevant proposals [28]. Probably, this lack of experimental results depends on a consistent strategy to correctly account for the peculiar coherent radiation propagation, pair production and thermal vibrations of the lattice. In case of optically active media, the parity breaking effects of ϵ_{R} are usually neglected in dealing with the laboratory data. *A fortiori*, no manifestation of time-reversal invariance was searched and then found, though, at present, many experiments are actively discussed to probe the tiny time-reversal violating manifestations. One of them is the observation of the rotation of the polarization plane in an atomic gas placed in an electric field, which manifests Macaluso–Corbino dispersive dependence of the deviation angle on the light frequency. This rotation is the kinematic analogue of the Faraday rotation in a magnetized field [23]. The violating effect arises due to the interaction of a non-zero electric dipole moment of the atom with the electric field [2].

In conclusion, we have proposed an attempt to describe the peculiar polarization invariances of the radiation propagation. The absorptive and dispersive contributions in a material can then resemble many intriguing features of high-energy particle physics. An important singularity of this paper is the inclusion of the absorptive effects. The resulting theory enlarges the standard approach based on the rotation group representations. It seems possible to describe these effects inside an abelian and strictly massless effective theory of the electromagnetic interactions, with a proper evolution of its transverse modes. In section 4, after the introduction of a shorthand notation in the preceeding sections, we described the propagation in linear birefringent media, with the individuation of their group invariants. In section 5, we considered the (elliptical) circular propagation in dichroic materials, and we proved the complications of a naive helicity description of the polarization observables. A strictly massless effective theory of the photon propagation can be assumed only at the expense of a sort of delocalization of (momentum-dependent) invariances. The aim of this study is to provide an incentive to consider properly parity and time reversal breaking observables in polarization optics.

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